Chapter 4

Stability

4.1 Autonomous systems

Now I switch to nonlinear systems. In this chapter the main object of study will be

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f}\colon X \longrightarrow \mathbf{R}^k, \tag{4.1}$$

where $\boldsymbol{f}$ is supposed to be locally Lipschitz in X. The maximal solution at the point t with the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ will be denoted usually as $\boldsymbol{x}(t; \boldsymbol{x}_0)$. Recall that the maximal solution through $\boldsymbol{x}_0$ does not have to be defined for all t and may exist on a shorter time interval $I(\boldsymbol{x}_0) = (t_-, t_+) \subseteq \mathbf{R}$.

Definition 4.1. *For each $\boldsymbol{x}_0 \in X$ the set*

$$\gamma(\boldsymbol{x}_0) = \{\boldsymbol{x}(t; \boldsymbol{x}_0)\colon t \in (t_-, t_+)\}$$

is called the orbit through $\boldsymbol{x}_0$. The set

$$\gamma_+(\boldsymbol{x}_0) = \{\boldsymbol{x}(t; \boldsymbol{x}_0)\colon t \in [0, t_+)\}$$

is called the positive semi-orbit through $\boldsymbol{x}_0$, and the set

$$\gamma_-(\boldsymbol{x}_0) = \{\boldsymbol{x}(t; \boldsymbol{x}_0)\colon t \in (t_-, 0]\}$$

is called the negative semi-orbit through $\boldsymbol{x}_0$. I have

$$\gamma(\boldsymbol{x}_0) = \gamma_-(\boldsymbol{x}_0) \cup \gamma_+(\boldsymbol{x}_0).$$

Positive semi-orbits are sometimes called *forward* orbits, and negative semi-orbits are sometimes called *backward* orbits. Orbits are the images of the solutions $t \mapsto \boldsymbol{x}(t; \boldsymbol{x}_0)$ to (4.1) parameterized by the time t such that the directions along the orbits are defined. The orbits should not be confused with the *integral curves* (the graphs of solutions in the extended phase space $I \times X$). It should be clear that the orbits are the projections of the integral curves onto the phase or state space X, and therefore carry less information than the solutions themselves. A number of examples of orbits were given when I discussed the phase portraits of the linear systems on the plane in the previous chapter.

Consider several important properties of the orbits of (4.1).

1. If $t \mapsto \boldsymbol{x}(t; \boldsymbol{x}_0)$ is a solution then $t \mapsto \boldsymbol{x}(t - t_0; \boldsymbol{x}_0)$ is also a solution for any constant t_0 (it was proved in Exercise 1.2, note that this fact is not true for non-autonomous systems). The orbits for these two solutions coincide, and the solutions are different by a translation by t_0 along the t-axis. (If the maximal solution for $t \mapsto \boldsymbol{x}(t; \boldsymbol{x}_0)$ is defined on $I(\boldsymbol{x}_0)$, what is the interval of existence of the maximal solution $t \mapsto \boldsymbol{x}(t - t_0; \boldsymbol{x}_0)$?)

2. Two orbits either do not intersect or coincide. To prove it consider two solutions $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ and assume that the corresponding orbits have a common points, i.e., there are t_1 and t_2 such that $\boldsymbol{\phi}_1(t_1) = \boldsymbol{\phi}_2(t_2)$. Consider also $\boldsymbol{\psi}(t) = \boldsymbol{\phi}_1(t + (t_1 - t_2))$. This is also a solution due to Property 1 with the same orbit as defined by $\boldsymbol{\phi}_1$. But $\boldsymbol{\psi}(t_2) = \boldsymbol{\phi}_2(t_2)$, i.e., due to the uniqueness theorem $\boldsymbol{\psi}$ coincides with $\boldsymbol{\phi}_2$ and therefore the orbits corresponding to $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ coincide.

There is a direct connection between orbits and integral curves. Here is a two dimensional example. Consider a planar autonomous system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2).$$

The orbits of this system are exactly the integral curves defined by

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)},$$

in those domains where $f_1(x_1, x_2) \neq 0$.

3. The simplest orbit is an *equilibrium*. By definition, $\hat{\boldsymbol{x}} \in X$ is an equilibrium of (4.1) if

$$\gamma(\hat{\boldsymbol{x}}) = \{\hat{\boldsymbol{x}}\},$$

i.e., if the corresponding orbit consists of just one point. The necessary and sufficient condition for $\hat{\boldsymbol{x}}$ to be an equilibrium is

$$\boldsymbol{f}(\hat{\boldsymbol{x}}) = 0.$$

Equilibria are also called *fixed points*, *stationary points*, or *rest points* of the dynamical system (4.1). Equilibria are the *critical* or *singular points* of the corresponding vector field $\boldsymbol{f}$, because at these points the vector field is undefined.

4. Recall that with every autonomous system (4.1) a dynamical system $\{X, I, \varphi^t\}$ is identified such that the group properties of the flow hold:

$$\boldsymbol{x}(0; \boldsymbol{x}_0) = \boldsymbol{x}_0,$$

and

$$\boldsymbol{x}(t_1 + t_2; \boldsymbol{x}_0) = \boldsymbol{x}(t_1; \boldsymbol{x}(t_2; \boldsymbol{x}_0)) = \boldsymbol{x}(t_2; \boldsymbol{x}(t_1; \boldsymbol{x}_0)), \quad t_1, t_2, t_1 + t_2 \in I(\boldsymbol{x}_0).$$

Obviously, in my notation, $\varphi^t \boldsymbol{x}_0 = \boldsymbol{x}(t; \boldsymbol{x}_0)$.

5. An orbit different from an equilibrium is a smooth curve. This follows directly from Property 3.

6. Every orbit either a smooth curve without self intersection, or a smooth closed curve (*cycle*), or a point. To each cycle corresponds a periodic solution.

To prove the last statement consider a cycle γ and take any point $\boldsymbol{x}_0$ on γ. Since the vector field never vanishes on γ, after some time T the solution starting at $\boldsymbol{x}_0$ will return to $\boldsymbol{x}_0$, i.e., $\boldsymbol{x}(T; \boldsymbol{x}_0) = \boldsymbol{x}_0$. Now take, for each fixed t, $\boldsymbol{x}(t + T; \boldsymbol{x}_0)$, which is also a solution by Property 1. By the group property I have $\boldsymbol{x}(t + T; \boldsymbol{x}_0) = \boldsymbol{x}(t; \boldsymbol{x}(T; \boldsymbol{x}_0)) = \boldsymbol{x}(t; \boldsymbol{x}_0)$, which proves that the solution corresponding to a cycle is T-periodic. The converse is trivial.

7. *Rectification of a vector field.*

Lemma 4.2. *Let $\boldsymbol{a} \in X$ be a non-equilibrium point of* (4.1)*. Then in a small neighborhood of $\boldsymbol{a}$ there exists a diffeomorphism $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{y})$ (i.e., a smooth change of the variables, the inverse to which is also smooth) such that in the new coordinates $\boldsymbol{y}$ the vector field looks like a family of straight lines:*

$$\dot{y}_1 = 0, \quad \ldots, \quad \dot{y}_{k-1} = 0, \quad \dot{y}_k = 1.$$

Proof. Without loss of generality I assume that $f_k(\boldsymbol{a}) \neq 0$. Consider a hyperplane in $\mathbf{R}^k\colon x_k = a_k$. Let me take an initial condition from this hyperplane: $\boldsymbol{x}_0 = (\xi_1, \ldots, \xi_{k-1}, a_k) = (\boldsymbol{\xi}, a_k)$. Now consider a solution $t \mapsto \boldsymbol{\phi}(t; \boldsymbol{\xi})$ to (4.1) such that $\boldsymbol{\phi}(0; \boldsymbol{\xi}) = (\boldsymbol{\xi}, a_k)$. I introduce new variables $y_1 = \xi_1, \ldots, y_{k-1} = \xi_{k-1}$, $y_k = t$. Obviously, $\dot{\boldsymbol{y}}$ defines a family of straight lines in $\mathbf{R}^k$. On the other hand I have that $\boldsymbol{x} = \boldsymbol{\phi}(t; \boldsymbol{\xi}) = \boldsymbol{\phi}(\boldsymbol{y})$, and I claim that this is the required change of variables. To finish the proof I need only to check that it is invertible, with smooth inverse such that $\boldsymbol{a} = \boldsymbol{\phi}(\boldsymbol{b})$, where $\boldsymbol{b} = (\xi_1, \ldots, \xi_{k-1}, 0)$. For this I compute the *Jacobi matrix* (see Section 2.9.1 if this term in not familiar) of $\boldsymbol{\phi}$ at $\boldsymbol{b}$ by using the fact that

$$\phi_i(0; \boldsymbol{\xi}) = \xi_i, \quad i = 1, \ldots, k-1,$$

and

$$\phi_k(0; \boldsymbol{\xi}) = a_k.$$

I find that

$$\frac{\partial \phi_i}{\partial y_j}(\boldsymbol{b}) = \delta_{ij}, \ \frac{\partial \phi_k}{\partial y_j} = 0, \quad i, j = 1, \ldots, k-1, \quad \frac{\partial \phi_k}{\partial y_k}(\boldsymbol{b}) = \frac{\partial \phi_k}{\partial t}(\boldsymbol{b}) = f_k(\boldsymbol{a}) \neq 0,$$

which completes the proof, because the Jacobi matrix by construction has non-zero determinant (which is equal to $f_k(\boldsymbol{a})$) and hence invertible by the *inverse function theorem* (Section 2.9.1). Here δ_{ij} is the Kronecker symbol, which is equal to 1 if $i = j$ and 0 otherwise. ■

Exercise 4.1. Rectify the vector field of

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1$$

in a neighborhood of $(x_1, x_2) = (1, 0)$. *Hint:* You are asked to find the change of the variables $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{y})$ such that in the $\boldsymbol{y}$ coordinates the vector field is composed of the straight lines, and also prove that this change of variables is invertible. Follow directly the proof of the lemma above.

Exercise 4.2. Find a diffeomorphism that rectifies the vector field of

$$\dot{x}_1 = x_1^2 x_2, \quad \dot{x}_2 = -x_1 x_2^2,$$

is a neighborhood of $(1, 1)$.

8. *Liouville's theorem.* Consider again (4.1), and let $\{\varphi^t\}$ denote the corresponding flow. Let D_0 be a bounded area in X and $m(D_0)$ be its measure (volume). By definition, $D_t := \varphi^t(D_0)$, and I am interested in how the measure of D_0 changes under the flow of (4.1). I use the convenient *nabla* or *del* operator $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right)$ to write the *divergence* of the function $\boldsymbol{f}$, in coordinates:

$$\operatorname{div} \boldsymbol{f} = \nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_k}{\partial x_k}\,.$$

Lemma 4.3. *Let* $V_t = m(D_t)$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}V_t\bigg|_{t=0} = \int_{D_0} \nabla\cdot\boldsymbol{f}\,\mathrm{d}\boldsymbol{x}.$$

Proof. I have that

$$V_t = \int_{D_t} 1\,\mathrm{d}\boldsymbol{x} = \int_{D_0} \det\left(\frac{\partial\varphi^t}{\partial\boldsymbol{x}}\right)\mathrm{d}\boldsymbol{x},$$

due to the change of the variables formula for multiple integrals. For the flow $\{\varphi^t\}$ Taylor's formula yields

$$\varphi^t\boldsymbol{x} = \boldsymbol{x} + t\boldsymbol{f}(\boldsymbol{x}) + \mathcal{O}(t^2),$$

and therefore the Jacobi matrix $\frac{\partial\varphi^t}{\partial\boldsymbol{x}}$ can be written as

$$\frac{\partial\varphi^t}{\partial\boldsymbol{x}} = \boldsymbol{I} + t\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}} + \mathcal{O}(t^2).$$

(I also used the notation $\boldsymbol{f'}(\boldsymbol{x})$ to denote the Jacobi matrix before.) Using the formula for the determinant $\det(\boldsymbol{I}+t\boldsymbol{A}) = 1 + t\operatorname{tr}\boldsymbol{A} + \mathcal{O}(t^2)$, I find

$$\det\left(\frac{\partial\varphi^t}{\partial\boldsymbol{x}}\right) = 1 + t\operatorname{tr}\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}} + \mathcal{O}(t^2).$$

Hence,

$$V_t = V_0 + \int_{D_0}\left(t\operatorname{tr}\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}} + \mathcal{O}(t^2)\right)\mathrm{d}\boldsymbol{x},$$

which proves the lemma, since

$$\operatorname{tr}\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}} = \nabla\cdot\boldsymbol{f} = \operatorname{div}\boldsymbol{f}.$$

■

Remark 4.4. A particular case of the previous lemma is Corollary 3.7.

Corollary 4.5. *If for* (4.1) $\nabla\cdot\boldsymbol{f}\equiv 0$ *then the corresponding flow conserves the phase volume.*

A vector field with the condition $\nabla\cdot\boldsymbol{f}\equiv 0$ is called the vector field without sources and sinks, because, due to the *Gauss theorem*, the flow of such vector field through any closed hypersurface S is equal to zero:

$$\int_S \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_D \nabla\cdot\boldsymbol{f}\,\mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{n}$ is the unit outward normal to the hypersurface S.

Example 4.6. Consider a *Hamiltonian system*

$$\begin{aligned}\dot{x}_i &= \frac{\partial H}{\partial p_i}(\boldsymbol{x},\boldsymbol{p}), \quad i=1,\ldots,k,\\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}(\boldsymbol{x},\boldsymbol{p}), \quad i=1,\ldots,k,\end{aligned}$$

where $H\in\mathcal{C}^{(1)}(\mathbf{R}^{2k};\mathbf{R}^{2k})$ is called the *Hamiltonian*. Using the previous I immediately get

Lemma 4.7 (Liouville's theorem)**.** *A Hamiltonian system conserves the phase volume.*

Exercise 4.3. Prove the lemma.

9. *First integrals* of (4.1).

Together with (4.1) consider a continuously differentiable function $u\colon X \longrightarrow \mathbf{R}$. If $\boldsymbol{\phi}$ is a solution to (4.1) then $u(\boldsymbol{\phi}(t)) = w(t)$ becomes a function of one variable t. Now consider the derivative

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \dot{u}(\boldsymbol{x}) = \sum_j \frac{\partial u}{\partial x_j}\dot{x}_j = \nabla u \cdot \boldsymbol{f}.$$

This derivative is called a *derivative of u along the vector field $\boldsymbol{f}$* (and is a generalization of the directional derivative, which takes into account only the direction and not the length of the vectors) or, sometimes, *Lie derivative*, after the Norwegian mathematician Sophus Lie (the last name is read "Lee"). Important to note that I sometimes can calculate it without knowing the actual solutions to (4.1).

Lemma 4.8. *Let $\dot{u}(\boldsymbol{x})$ be the derivative along the vector field* (4.1), *and $\dot{u}(\boldsymbol{x}) \leq 0$ ($\dot{u}(\boldsymbol{x}) \geq 0$) in some $U \subseteq X$. Then $u(\boldsymbol{x})$ does not increase* (*decrease*) *along any orbit of* (4.1) *in U.*

Exercise 4.4. Prove this lemma.

Definition 4.9. *Function u is called a first integral of* (4.1) *if it is constant along any orbit of* (4.1).

Lemma 4.10. *Function u is a first integral of* (4.1) *if and only if*

$$\dot{u}(\boldsymbol{x}) = 0$$

along the vector field (4.1).

Exercise 4.5. Prove this lemma.

Let me say a few words about the geometric interpretation of the condition

$$\dot{u}(\boldsymbol{x}) = \nabla u \cdot \boldsymbol{f} = 0.$$

The vector ∇u (recall that it is called the *gradient*) is orthogonal to the hypersurface $S\colon u(\boldsymbol{x}) = \text{const}$, which means that vector $\boldsymbol{f}$ is tangent to S, and therefore an orbit passing through $\boldsymbol{x} \in S$ stays on S, which means that $u(\boldsymbol{x}) \equiv \text{const}$ on this orbit.

Example 4.11. For a Hamiltonian system the Hamiltonian H is a first integral.

Example 4.12. The movement of a particle with one degree of freedom in the potential field is described by Newton's equation

$$m\ddot{x} = -U'(x).$$

The function

$$\frac{m\dot{x}^2}{2} + U(x) = E$$

is a first integral (check this) and represents the full energy of the system (the sum of the kinetic and potential energies).

Exercise 4.6. Find all independent first integrals of

$$\dot{x}_1 = 0, \quad \dots, \quad \dot{x}_{k-1} = 0, \quad \dot{x}_k = 1.$$

(Function $u_1(\boldsymbol{x}), \dots, u_m(\boldsymbol{x})$, $m \leq k$ are called independent in U if none of them can be expressed through the others.)

Exercise 4.7. Prove that in a neighborhood of a point $\boldsymbol{a}$ such that $\boldsymbol{f}(\boldsymbol{a}) \neq 0$ there exist $k-1$ independent first integrals of (4.1).

4.2 Lyapunov stability (second Lyapunov method)

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f} \colon X \longrightarrow \mathbf{R}^k, \tag{4.2}$$

and assume that $\hat{\boldsymbol{x}}$ is such that $\boldsymbol{f}(\hat{\boldsymbol{x}}) = 0$, i.e., $\hat{\boldsymbol{x}}$ is an equilibrium.

Definition 4.13. *Equilibrium $\hat{\boldsymbol{x}}$ is called Lyapunov stable if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$|\boldsymbol{x}(t; \boldsymbol{x}_0) - \hat{\boldsymbol{x}}| < \epsilon, \quad t > 0,$$

whenever

$$|\boldsymbol{x}_0 - \hat{\boldsymbol{x}}| < \delta.$$

Equilibrium $\hat{\boldsymbol{x}}$ is called asymptotically stable if it is Lyapunov stable and, additionally,

$$|\boldsymbol{x}(t; \boldsymbol{x}_0) - \hat{\boldsymbol{x}}| \to 0, \quad t \to \infty.$$

Otherwise, $\hat{\boldsymbol{x}}$ is called unstable.

Exercise 4.8. Formulate an ϵ–δ definition of an unstable equilibrium.

Example 4.14. Recall that in the case of the linear system with constant coefficients $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{x}(t) \in \mathbf{R}^2$, I classified different phase portraits as sinks (stables nodes and foci), sources (unstable nodes and foci), saddles, and centers. Naturally I have that the sinks are asymptotically stable, the centers are Lyapunov stable but not asymptotically stable, sources and saddles are unstable. Moreover, the terminology stable, asymptotically stable, and unstable linear systems implies that the stability of the whole system coincides with the stability of the trivial equilibrium $(0, 0)$ of this system.

Exercise 4.9. Prove that if any solution of a linear homogeneous system of ODE with constant coefficients is bounded for $t \to \infty$ then the trivial solution is Lyapunov stable.

Example 4.15. For a scalar differential equation

$$\dot{x} = f(x), \quad x(t) \in X \subseteq \mathbf{R},$$

the fact that $f'(\hat{x}) < 0$ implies asymptotical stability of $\hat{x}$ and $f'(\hat{x}) > 0$ implies instability of $\hat{x}$.

Exercise 4.10. Consider a scalar differential equation $\dot{x} = f(x)$, $x(t) \in X \subseteq \mathbf{R}$ and assume that $\hat{x}$ is an equilibrium. Prove that if $f'(\hat{x}) < 0$ then $\hat{x}$ is asymptotically stable, and if $f'(\hat{x}) > 0$ then $\hat{x}$ is unstable. Can you determine stability if $f'(\hat{x}) = 0$? Consider the equation $\dot{x} = rx(1 - x/K)$, $r, K > 0$, find the equilibria and determine their stability.

I will be studying the stability of equilibria using the so-called *Lyapunov functions*, but first I will need several auxiliary facts. First, a smooth function $V\colon U\subseteq\mathbf{R}^k\longrightarrow\mathbf{R}$ is called *positive definite* in U if $V(\boldsymbol{x})>0$ for $\boldsymbol{x}\in U$, $\boldsymbol{x}\neq\hat{\boldsymbol{x}}$ and $V(\hat{\boldsymbol{x}})=0$. For example, $V(\boldsymbol{x})=\sum_j x_j^2$ is positive definite in any neighborhood of the origin. If $V(\boldsymbol{x})<0$ in $U\backslash\{\hat{\boldsymbol{x}}\}$ and $V(\hat{\boldsymbol{x}})=0$ then V is called *negative definite.* If the conditions >0 or <0 are replaced with ≥ 0 and ≤ 0, then V is called positive of negative *semi-definite* (or non-negative and non-positive definite).

Example 4.16. Consider the quadratic form

$$V(\boldsymbol{x})=\boldsymbol{A}\boldsymbol{x}\cdot\boldsymbol{x}=\sum_{i,j}a_{ij}x_ix_j,$$

where $a_{jk}=a_{kj}$ such that $\boldsymbol{A}=\boldsymbol{A}^\top$ is a symmetric real matrix. This quadratic form is called positive definite if

$$\boldsymbol{A}\boldsymbol{x}\cdot\boldsymbol{x}>0,\quad \boldsymbol{x}\neq 0.$$

A necessary and sufficient condition for this quadratic form to be positive definite is to have all the eigenvalues of $\boldsymbol{A}$ positive, which I prove next.

Lemma 4.17. *If $\boldsymbol{A}=\boldsymbol{A}^\top$ is a real matrix, then for any $\boldsymbol{x}\in\mathbf{R}^k$*

$$\alpha|\boldsymbol{x}|^2\leq\boldsymbol{A}\boldsymbol{x}\cdot\boldsymbol{x}\leq\beta|\boldsymbol{x}|^2,$$

where α and β are the minimal and maximal eigenvalues of $\boldsymbol{A}$.

Proof. I will use the fact that if $\boldsymbol{A}$ is real symmetric then there exists an orthogonal matrix $\boldsymbol{T}$ such that $\boldsymbol{T}\boldsymbol{T}^\top=\boldsymbol{I}$ and

$$\boldsymbol{T}^\top\boldsymbol{A}\boldsymbol{T}=\boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}=\operatorname{diag}(\lambda_1,\ldots,\lambda_k)$, and λ_j, $j=1,\ldots,k$ are the real eigenvalues of $\boldsymbol{A}$.

I have, putting $\boldsymbol{x}=\boldsymbol{T}\boldsymbol{y}$,

$$\begin{aligned}\boldsymbol{A}\boldsymbol{x}\cdot\boldsymbol{x}&=\boldsymbol{A}\boldsymbol{T}\boldsymbol{y}\cdot\boldsymbol{T}\boldsymbol{y}=\boldsymbol{T}^\top\boldsymbol{A}\boldsymbol{T}\boldsymbol{y}\cdot\boldsymbol{y}=\\&=\boldsymbol{\Lambda}\boldsymbol{y}\cdot\boldsymbol{y}=\sum_{j=1}^k\lambda_jy_j^2,\end{aligned}$$

such that

$$\alpha|\boldsymbol{y}|^2\leq\boldsymbol{A}\boldsymbol{x}\cdot\boldsymbol{x}\leq\beta|\boldsymbol{y}|^2.$$

Since the orthogonal transformation preserves the length: $|\boldsymbol{x}|^2=|\boldsymbol{T}\boldsymbol{y}|^2=|\boldsymbol{y}|^2$, this concludes the proof. ∎

Definition 4.18. *A positive definite in a neighborhood U of $\hat{\boldsymbol{x}}$ function V is called Lyapunov function for $\hat{\boldsymbol{x}}$ if*

$$\dot V(\boldsymbol{x})\leq 0,\quad \boldsymbol{x}\in U.$$

It is called a strict Lyapunov function for $\hat{\boldsymbol{x}}$ if

$$\dot V(\boldsymbol{x})<0,\quad \boldsymbol{x}\in U,\quad \boldsymbol{x}\neq\hat{\boldsymbol{x}}.$$

Theorem 4.19 (Lyapunov)**.** *Consider* (4.2) *and let* $\hat{\boldsymbol{x}}$ *be an equilibrium. Then* $\hat{\boldsymbol{x}}$ *is Lyapunov stable if there exists a Lyapunov function for* $\hat{\boldsymbol{x}}$ *and asymptotically stable if there exists a strict Lyapunov function for* $\hat{\boldsymbol{x}}$*.*

Proof. Without loss of generality assume that $\hat{\boldsymbol{x}} = 0$ and pick $\epsilon > 0$ such that the ball $B_\epsilon\colon |\boldsymbol{x}| \leq \epsilon \subseteq U$. Let S_ϵ be the boundary of B_ϵ. Since S_ϵ is compact, V is continuous, $V(\boldsymbol{x}) > 0$ on S_ϵ then $\min_{\boldsymbol{x}\in S_\epsilon} V(\boldsymbol{x}) = \beta > 0$. Consider another ball $B_\delta\colon |\boldsymbol{x}| \leq \delta \subseteq U$. Since $V(0) = 0$, I can always choose $\delta > 0$ such that $V(\boldsymbol{x}) < \beta$ for $\boldsymbol{x} \in B_\delta$. I need to show that if $|\boldsymbol{x}_0| \leq \delta$ then $|\boldsymbol{x}(t;\boldsymbol{x}_0)| \leq \epsilon$ for $t > 0$. Since $\dot{V}(\boldsymbol{x}) \leq 0$ then $V(\boldsymbol{x}_0) < \beta$ implies that $V(\boldsymbol{x}) < \beta$ on the positive orbit $\boldsymbol{x}(t;\boldsymbol{x}_0)$, $t > 0$. Therefore, the orbit starting in B_δ cannot cross the boundary of B_ϵ because $V(\boldsymbol{x}) \geq \beta$ on S_ϵ.

To prove the asymptotic stability, I pick the same balls B_ϵ and B_δ and consider $w(t) = V(\boldsymbol{x}(t;\boldsymbol{x}_0))$. Since $\dot{V} \leq 0$ then $w(t)$ is non-increasing and bounded, and hence there exists a limit A. If $A = 0$ then nothing to prove. Assume that $A > 0$. I have $w(t) \geq A$ for all $t > 0$ and therefore there is $\alpha > 0$ such that $|\boldsymbol{x}(t;\boldsymbol{x}_0)| \geq \alpha$ because otherwise the orbit from $\boldsymbol{x}_0$ would be close to zero and hence $V(\boldsymbol{x})$ would be close to zero. In the set $\alpha \leq |\boldsymbol{x}| \leq \epsilon$ by the assumption I have that $\dot{V}(\boldsymbol{x}) \leq -m < 0$, and therefore $w'(t) \leq -m$. Integrating yields $w(t) \leq w(0) - mt$, which becomes negative for sufficiently large t, which contradicts $w(t) \geq 0$. ■

It is almost immediate to prove a converse of Lyapunov's theorem: If a positive definite function V has a positive semi-definite derivative along the vector field $\boldsymbol{f}$, than the equilibrium is unstable (do this). However, in specific situations it is usually impossible to find such V. Note also that for an equilibrium to be unstable is it enough to have just one initial condition $\boldsymbol{x}_0$ close to $\hat{\boldsymbol{x}}$ such that $\boldsymbol{x}(t;\boldsymbol{x}_0)$ leaves a neighborhood of $\hat{\boldsymbol{u}}$ for $t > 0$. This allows to find a better criterion for an equilibrium to be unstable. This criterion is sometimes called the *Chetaev instability theorem.*

Theorem 4.20 (Chetaev)**.** *Let* U *be a neighborhood of* $\hat{\boldsymbol{x}}$*,* $U_1 \subset U$*, and* $\hat{\boldsymbol{x}} \in \partial U_1$*, boundary of* U_1*. Let* $V \in \mathcal{C}^{(1)}(U_1;\mathbf{R})$ *be such that*

$$V(\boldsymbol{x}) > 0, \quad \dot{V}(\boldsymbol{x}) > 0, \quad \boldsymbol{x} \in U_1,$$

and $V(\boldsymbol{x}) = 0$ *at those boundary points of* U_1 *that lie inside* U*. Then* $\hat{\boldsymbol{x}}$ *is unstable.*

Proof. Consider a positive orbit $\boldsymbol{x}(t;\boldsymbol{x}_0)$, $t > 0$, $\boldsymbol{x}_0 \in U_1$, and finction $w(t) = V(\boldsymbol{x}(t;\boldsymbol{x}_0))$ along this orbit. I have $w(0) > 0$ and $w'(t) = \dot{V}(\boldsymbol{x}) > 0$ and therefore this orbit cannot cross the boundary of U_1 where $V(\boldsymbol{x}) = 0$ and eventually must leave U_1. Since U_1 has points arbitrary close to $\hat{\boldsymbol{x}}$, therefore this equilibrium is unstable. ■

Remark 4.21. There are no universal methods to find either Lyapunov or Chetaev functions (although often special form of a system of ODE hints for a form of Lyapunov function, there are books devoted these special cases), so it is a good idea to start with something simple like $ax_1^2 + bx_2^2$ or $ax_1^4 + bx_2^4$ and so on. Often a positive definite quadratic form for an appropriate matrix $\boldsymbol{A}$ is a good choice.

Exercise 4.11. Study the stability properties of the trivial solution in the following problems:

1. (Stable)

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1 + x_1x_2,\\ \dot{x}_2 &= x_1 - x_2 - x_1^2 - x_2^3.\end{aligned}$$

2. (Stable)

$$\dot{x}_1 = 2x_2^3 - x_1^5,$$
$$\dot{x}_2 = -x_1 - x_2^3 + x_2^5.$$

3. (Unstable, try $V(\boldsymbol{x}) = x_1x_2$)

$$\dot{x}_1 = x_1x_2 - x_1^3 + x_2^3,$$
$$\dot{x}_2 = x_1^2 - x_2^2.$$

4. (Asymptotically stable, try $V(\boldsymbol{x}) = ax_1^2 + bx_2^2$ and determine a, b)

$$\dot{x}_1 = x_1x_2^2 - \frac{1}{2}x_1^3,$$
$$\dot{x}_2 = -\frac{1}{2}x_2^3 + \frac{1}{5}x_1^2x_2.$$

5. (Unstable, try $V(\boldsymbol{x}) = x_2^2 - x_1^2$)

$$\dot{x}_1 = -x_1 - x_1x_2,$$
$$\dot{x}_2 = -x_2^3 - x_1^3.$$

6.

$$\dot{x}_1 = -x_1x_2^4,$$
$$\dot{x}_2 = x_2x_1^4.$$

7.

$$\dot{x}_1 = x_1 - x_1x_2^4,$$
$$\dot{x}_2 = x_2 - x_1^2x_2^3.$$

Exercise 4.12. Determine the stability properties of $(x, \dot{x}) = (0, 0)$ for the equation

$$\ddot{x} + x^n = 0, \quad n \in \mathbf{N}.$$

4.3 Stability of linear systems

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t) \in \mathbf{R}^k, \tag{4.3}$$

with a real matrix $\boldsymbol{A}$. I already proved in the previous chapter that the trivial equilibrium $\hat{\boldsymbol{x}} = 0$ of this system is asymptotically stable if and only if the real parts of all the eigenvalues of $\boldsymbol{A}$ are negative. Let me prove this fact again using a Lyapunov function. First, I will present an auxiliary fact that any matrix can be put in an almost diagonal form.

Theorem 4.22. *Any matrix $\boldsymbol{A}$ can be represented as*

$$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \boldsymbol{\Lambda} + \boldsymbol{B}_\epsilon,$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with the eigenvalues of $\boldsymbol{A}$ on the main diagonal, and matrix $\boldsymbol{B}_\epsilon$ has the entries $|b_{ij}| < \epsilon$, where $\epsilon > 0$ can be chosen arbitrarily small.

Proof. Any matrix $\boldsymbol{A}$ can be put in the Jordan canonical form

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{J},$$

where $\boldsymbol{J} = \operatorname{diag}(\boldsymbol{J}_1, \ldots, \boldsymbol{J}_s)$. For each Jordan's block $\boldsymbol{J}_\alpha$ consider the matrix of the same size $\boldsymbol{R}_\alpha = \operatorname{diag}(a_1, \ldots, a_l)$, then

$$\boldsymbol{R}^{-1}\boldsymbol{J}\boldsymbol{R} = \operatorname{diag}(\lambda, \ldots, \lambda) + \begin{bmatrix} 0 & a_2 a_1^{-1} & & 0 \\ & 0 & a_3 a_2^{-1} & 0 \\ & & \ddots & a_l a_{l-1}^{-1} \\ & & & 0 \end{bmatrix}.$$

Choose $a_2 a_1^{-1} = \ldots = a_l a_{l-1}^{-1} = \epsilon$.

If $\boldsymbol{R} = \operatorname{diag}(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_s)$ then

$$\boldsymbol{R}^{-1}\boldsymbol{J}\boldsymbol{R} = \boldsymbol{\Lambda} + \boldsymbol{B}_\epsilon.$$

Now take $\boldsymbol{T} = \boldsymbol{P}\boldsymbol{R}$, which concludes the proof. ■

Theorem 4.23. *The equilibrium $\hat{\boldsymbol{x}} = 0$ of (4.3) is asymptotically stable if and only if $\operatorname{Re}\lambda_j < 0$ for all $j = 1, \ldots, k$, where λ_j are the eigenvalues of $\boldsymbol{A}$.*

Proof. By using $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$ from the previous theorem, system (4.3) becomes

$$\dot{\boldsymbol{y}} = (\boldsymbol{\Lambda} + \boldsymbol{B}_\epsilon)\boldsymbol{y}.$$

Consider

$$V(\boldsymbol{x}) = \boldsymbol{y} \cdot \overline{\boldsymbol{y}} = \sum_{j=1}^{k} |y_j|^2.$$

This function is positive definite in any neighborhood of 0. I have

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{y} \cdot \overline{\boldsymbol{y}}) = \dot{\boldsymbol{y}} \cdot \overline{\boldsymbol{y}} + \boldsymbol{y} \cdot \dot{\overline{\boldsymbol{y}}} = \\ &= (\boldsymbol{\Lambda} + \overline{\boldsymbol{\Lambda}})\boldsymbol{y} \cdot \overline{\boldsymbol{y}} + \boldsymbol{B}_\epsilon \boldsymbol{y} \cdot \overline{\boldsymbol{y}} + \boldsymbol{y} \cdot \overline{\boldsymbol{B}}_\epsilon \overline{\boldsymbol{y}} \\ &\leq -2(\alpha - k\epsilon) \sum_{j=1}^{k} |y_j|^2 = -2(\alpha - k\epsilon) V(\boldsymbol{x}). \end{aligned}$$

Here I used

$$\sum_{j=1}^{k} (\lambda_j + \overline{\lambda}_j)|y_j|^2 = 2\sum_{j=1}^{k} \operatorname{Re}\lambda_j |y_j|^2 \leq -2\alpha V(\boldsymbol{x}), \quad \operatorname{Re}\lambda_j \leq -\alpha,$$

and

$$|\boldsymbol{B}_\epsilon \boldsymbol{y}\cdot\overline{\boldsymbol{y}}| \leq \epsilon\sum_{i,j=1}^{k}|y_i||y_j| = \epsilon\left(\sum_{j=1}^{k}|y_j|\right)^2 \leq k\epsilon\sum_{j=1}^{k}|y_j|^2.$$

The same estimate holds for the last term. Pick $0<\epsilon<\alpha/k$, then $\dot{V}$ is negative definite.

To finish the proof, I can assume that $\operatorname{Re}\lambda_j \geq 0$ for some j. Then the solution $e^{\lambda_j t}\boldsymbol{v}_j$, where $\boldsymbol{v}_j$ is the corresponding eigenvector, does not tend to zero, and therefore the trivial solution is not asymptotically stable. To practice Chetaev's theorem, I will give an alternative proof of this fact.

Let λ_k have positive real part. I can always find a matrix $\boldsymbol{T}$ that puts $\boldsymbol{A}$ in a triangular form $\boldsymbol{\Lambda}$ such that the last diagonal element is λ_k, i.e., I have the differential equation

$$\dot{y}_k = \lambda_k y_k.$$

Let me take the Chetaev function in the form

$$V(\boldsymbol{x}) = |y_k|^2.$$

I have

$$\dot{V}(\boldsymbol{x}) = \dot{y}_k\cdot\overline{y}_k + y_k\cdot\dot{\overline{y}}_k = 2\operatorname{Re}\lambda_k|y_k|^2.$$

As U_1 let me take $y_k\neq 0$. Since $\boldsymbol{y}=\boldsymbol{T}^{-1}\boldsymbol{x}$, then $y_k=\sum_j c_j x_j$, where c_j are some complex constants, and the boundary of U_1 is defined by

$$\sum_{j=1}^{k} x_j\operatorname{Re}c_j = 0,\quad \sum_{j=1}^{k} x_j\operatorname{Im}c_j = 0.$$

Hence ∂U_1 contains 0, and $V(\boldsymbol{x})>0$ and $\dot{V}(\boldsymbol{x})>0$ for $\boldsymbol{x}\in U_1$, $V(\boldsymbol{x})=0$ on ∂U_1, then all the conditions of the Chetaev theorem are satisfied and I proved that the origin is unstable. ■

Actually, I showed even more than it was stated in the theorem. I also showed that

$$\dot{V}(\boldsymbol{x}) \leq -2\beta V(\boldsymbol{x}),\quad \beta=\alpha-k\epsilon.$$

Since $V(\boldsymbol{x})$ is positive definite and can be written as a quadratic form

$$V(\boldsymbol{x}) = \sum_{i,j=1}^{k} a_{ij}x_i x_j,$$

then there exists a $c>0$ such that $V(\boldsymbol{x})\geq c|\boldsymbol{x}|^2$ (see Lemma 4.17). These two fact together imply that

$$|\boldsymbol{x}(t;\boldsymbol{x}_0)| \leq Ce^{-\beta t},\quad C>0,$$

i.e., that the convergence to the equilibrium is *exponential.*

Exercise 4.13. Prove

Lemma 4.24. *Let $\hat{\boldsymbol{x}} = 0$ be an equilibrium with the Lyapunov function V such that*

$$\dot{V}(\boldsymbol{x}) \leq -\gamma V(\boldsymbol{x}), \quad V(\boldsymbol{x}) \geq A|\boldsymbol{x}|^{\eta},$$

in some neighborhood U of 0. Here A, γ, η are positive constants. Then there exists $C > 0$ such that

$$|\boldsymbol{x}(t; \boldsymbol{x}_0)| \leq C e^{-\gamma t/\eta}, \quad t > 0,$$

if $\boldsymbol{x}_0$ is sufficiently close to 0.

All of the results in this section were already proved by using the explicit form of the solution $e^{t\boldsymbol{A}}\boldsymbol{x}_0$ to the linear system. However, the value of this approach in that it can be transferred without much change onto nonlinear systems, see the next section.

4.4 Stability of equilibria of nonlinear systems by linearization (first Lyapunov method)

Consider now

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f} \colon X \longrightarrow \mathbf{R}^k, \tag{4.4}$$

and let $\hat{\boldsymbol{x}}$ be an equilibrium. Assuming $\boldsymbol{f} \in \mathcal{C}^{(1)}$ I can represent it with Taylor's formula around $\hat{\boldsymbol{x}}$

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}}) + \mathcal{O}(|\boldsymbol{x} - \hat{\boldsymbol{x}}|^2) = \boldsymbol{A}(\boldsymbol{x} - \hat{\boldsymbol{x}}) + \mathcal{O}(|\boldsymbol{x} - \hat{\boldsymbol{x}}|^2).$$

Here

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \dots & \frac{\partial f_k}{\partial x_k} \end{bmatrix}$$

is the Jacobi matrix of $\boldsymbol{f}$.

Using the change of variables $\boldsymbol{y} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ and dropping the terms of the order $o(|\boldsymbol{x} - \hat{\boldsymbol{x}}|)$ I end up with the linear system

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{y}, \tag{4.5}$$

which is called the *linearization* of (4.4) around $\hat{\boldsymbol{x}}$. I know that the stability of the trivial solution to (4.5) is determined by the eigenvalues of matrix $\boldsymbol{A}$. It turns out that the eigenvalues of $\boldsymbol{A}$ also allow to infer the stability of $\hat{\boldsymbol{x}}$ in some cases.

Theorem 4.25 (Lyapunov)**.** *If the origin of the linearized system is asymptotically stable then $\hat{\boldsymbol{x}}$ is also asymptotically stable. If the Jacobi matrix has at least one eigenvalue with $\operatorname{Re}\lambda_j > 0$, then $\hat{\boldsymbol{x}}$ is unstable.*

Proof. Using the notations as above and assuming, without loss of generality, that $\hat{\boldsymbol{x}} = 0$, I have

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}), \quad |\boldsymbol{g}(\boldsymbol{x})| \leq C_1|\boldsymbol{x}|^2.$$

As a Lyapunov function I take exactly the same V, which was used for the linear system in the previous section. After the change of variables $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$ I have

$$\dot{\boldsymbol{y}} = (\boldsymbol{\Lambda} + \boldsymbol{B}_\epsilon)\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}), \quad \boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{T}^{-1}\boldsymbol{g}(\boldsymbol{T}\boldsymbol{y}).$$

Calculating $\dot{V}$ yields

$$\dot{V}(\boldsymbol{x}) = A_1 + A_2,$$

where $A_1 \leq -\gamma|\boldsymbol{x}|^2$, $\gamma > 0$ is from the linear system, and

$$A_2 = \boldsymbol{h}(\boldsymbol{y}) \cdot \overline{\boldsymbol{y}} + \boldsymbol{y} \cdot \overline{\boldsymbol{h}}(\boldsymbol{y}).$$

I have that

$$A_2 \leq C_3|\boldsymbol{x}|^3,$$

since $|\boldsymbol{y}| \leq C_2|\boldsymbol{x}|$ and $|\boldsymbol{h}(\boldsymbol{y})| \leq C_1C_2|\boldsymbol{x}|^2$. This implies that

$$\dot{V}(\boldsymbol{x}) \leq -|\boldsymbol{x}|^2(\gamma - C_3|\boldsymbol{x}|).$$

If $|\boldsymbol{x}| < \gamma/(2C_3)$, then

$$\dot{V}(\boldsymbol{x}) \leq -\frac{\gamma}{2}|\boldsymbol{x}|^2,$$

which proves the asymptotical stability of $\hat{\boldsymbol{x}}$. Moreover, I showed that the convergence is exponential (the details are left to the reader).

To prove the second statement of the theorem, assume that $\operatorname{Re}\lambda_k > 0$ and take $V(\boldsymbol{x}) = |y_k|^2$ (see the previous section for the details). Then,

$$V(0) = 0, \quad \dot{V}(\boldsymbol{x}) = 2\operatorname{Re}\lambda_k|y_k|^2 + \boldsymbol{h}(\boldsymbol{y}),$$

where $|\boldsymbol{h}(\boldsymbol{y})| \leq C|\boldsymbol{y}|^3$. I have

$$\dot{V}(\boldsymbol{x}) \geq (2\operatorname{Re}\lambda_k - C|y_k|)|y_k|^2,$$

at the point $y_1 = \ldots = y_{k-1} = 0$, which shows that $V(\boldsymbol{x})$ is a Chetaev function and hence the origin is unstable. ■

This important relation between linear and nonlinear systems is a first results in a number of deep connections (see also Appendix to this chapter).

Definition 4.26. *An equilibrium point $\hat{\boldsymbol{x}}$ is called hyperbolic if the Jacobi matrix evaluated at this point has no eigenvalues with zero real part.*

Using this definition Lyapunov's theorem from this section can be restated as

Theorem 4.27. *Stability of a hyperbolic equilibrium coincides with the stability of its linearization.*

Exercise 4.14. For which α the system

$$\begin{aligned}\dot{x}_1 &= x_2 - \alpha x_1 - x_1^5,\\ \dot{x}_2 &= -x_1 - x_2^5,\end{aligned}$$

has a stable equilibrium at the origin?

Exercise 4.15. Consider

$$\begin{aligned}\dot{x}_1 &= -x_1 - \frac{x_2}{\log(x_1^2 + x_2^2)^{1/2}},\\ \dot{x}_2 &= -x_2 + \frac{x_1}{\log(x_1^2 + x_2^2)^{1/2}}.\end{aligned}$$

Show that in full nonlinear system the origin is a spiral, whereas it is a node in linearization. *Hint:* use the polar coordinates.

Exercise 4.16. Can an asymptotically stable equilibrium become unstable in Lyapunov's sense under linearization?

4.5 More on the notion of stability

Up till now I discussed the stability properties of equilibria. Recall that the equilibria correspond to the constant solutions to the original system. Very naturally, the definitions of asymptotic stability or Lyapunov stability can be generalized for an arbitrary solution to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}). \tag{4.6}$$

To wit, a solution $\boldsymbol{\phi}$ to (4.6) is called *Lyapunov stable* if for any $\epsilon > 0$ and t_0 there exists a $\delta(\epsilon, t_0) > 0$ such that

$$|\boldsymbol{\phi}(t) - \boldsymbol{x}(t; t_0, \boldsymbol{x}_0)| < \epsilon, \quad t > 0,$$

whenever

$$|\boldsymbol{\phi}(t_0) - \boldsymbol{x}_0| < \delta.$$

Here $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ is the solution to (4.6) with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

Solution $\boldsymbol{\phi}$ to (4.6) is called *asymptotically stable* if it is Lyapunov stable, and, additionally

$$|\boldsymbol{\phi}(t) - \boldsymbol{x}(t; t_0, \boldsymbol{x}_0)| \to 0, \quad t \to \infty.$$

Example 4.28. Each solution to $\dot{x} = 0$ is Lyapunov stable but not asymptotically stable.

Example 4.29. Each solution to $\dot{x} + x = 0$ is asymptotically stable. Indeed, the general solution is $x(t) = Ce^{-t}$, and for two initial conditions x_1^0, x_2^0 I have

$$x_1(t) = x_1^0 e^{-(t-t_0)}, \quad x_2(t) = x_2^0 e^{-(t-t_0)},$$

from where

$$|x_2(t) - x_1(t)| = |x_1^0 - x_2^0| e^{-(t-t_0)} \to 0.$$

Exercise 4.17. Is the solution to

$$\dot{x} = 4x - t^2 x, \quad x(0) = 0$$

Lyapunov stable, asymptotically stable, or neither?

Analysis of stability of an arbitrary solution $\boldsymbol{\phi}$ to (4.6) can be reduced to the analysis of the trivial solution of some new system, which is obtained from (4.6) by the linear change $\boldsymbol{x}(t) = \boldsymbol{y}(t) + \boldsymbol{\phi}(t)$. For the new variable I find

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y} + \boldsymbol{\phi}) - f(t, \boldsymbol{\phi}) = \boldsymbol{g}(t, \boldsymbol{y}), \tag{4.7}$$

which clearly has the trivial solution $\boldsymbol{y}(t) = 0$ for all t. Moreover, the type of stability of ϕ coincides with the type of stability of the trivial solution $\boldsymbol{y}(t) = 0$.

I can also consider a linearization of (4.7) around $\boldsymbol{y}(t) = 0$, which in general takes the form

$$\dot{\boldsymbol{y}} = \boldsymbol{A}(t)\boldsymbol{y},$$

with a non-constant $\boldsymbol{A}(t)$. As it was mentioned before, I cannot make any conclusions by the eigenvalues of $\boldsymbol{A}(t)$ if it is time dependent. If it is, however, constant, I can prove a very similar fact: if the matrix of linearization is constant and has all the eigenvalues with $\operatorname{Re} \lambda_j < 0$ then the trivial solution of the original problem is asymptotically stable; if there is at least one eigenvalue with $\operatorname{Re} \lambda_j > 0$ then the trivial solution of the original problem is unstable.

Example 4.30. Study the stability of the periodic solution to

$$\dddot{x} + x = \cos t.$$

The general solution to this equation, obtained in the usual fashion, is given by

$$x(t) = C_1 e^{-t} + e^{\frac{t}{2}}(C_2 \cos \frac{\sqrt{3}}{2} t + C_3 \sin \frac{\sqrt{3}}{2} t) + \frac{1}{2}(\cos t - \sin t),$$

therefore the only periodic solution is

$$\phi(t) = \frac{1}{2}(\cos t - \sin t).$$

Let

$$x(t) = \phi(t) + y(t).$$

I get

$$\dddot{y} + y = 0,$$

and the trivial solution has eigenvalues $-1, (1 \pm \mathrm{i}\sqrt{3})/2$, and therefore the trivial solution is unstable, which implies the instability of the original periodic solution.

Exercise 4.18. Is the π-periodic solution to the system

$$x' = x - y, \quad y' = 2x - y + 6\sin^2 t$$

Lyapunov stable?

Exercise 4.19. Consider

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{f}(t),$$

with $\boldsymbol{A}, \boldsymbol{f}$ continuous on $\mathbf{R}$. Prove that if one solution to this problem is stable (asymptotically stable) then any solution is stable (asymptotically stable).

Exercise 4.20. Decide whether the solution $x_1(t) = \cos t$, $x_2(t) = 2\sin t$ to the system

$$\begin{aligned} \dot{x}_1 &= \ln\left(x_1 + 2\sin^2 \frac{t}{2}\right) - \frac{x_2}{2}, \\ \dot{x}_2 &= (4 - x_1^2)\cos t - 2x_1 \sin^2 t - \cos^3 t, \end{aligned}$$

stable or not.

Exercise 4.21. Let ϕ be a T-periodic solution to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Show that the linearization of this system around $\boldsymbol{\phi}$ is a linear non-autonomous system with a T-periodic matrix.

4.6 Limit sets and Lyapunov functions

4.6.1 Analysis of the pendulum equation

I am going to start with a motivating example.

Example 4.31. Consider the pendulum equation

$$\ddot{\theta} + k \sin \theta = 0,$$

where θ is the angle of the pendulum rod from the vertical and $k > 0$ is a constant, $k = g/l$, where $g \approx 10 m/sec^2$, and l in the length of the rod. It is assumed that there are no damping or external forces acting on the pendulum. For simplicity assume that $k = 1$. In the coordinates $(x, y) = (\theta, \dot{\theta})$, I have the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sin x. \end{aligned}$$

The equilibria of this system are given by $\hat{y} = 0$ and $\hat{x}_j = j\pi$, $n = 0, \pm 1, \pm 2, \ldots$. (This is true if I consider this system on the state space $\mathbf{R}^2$, but since $x = \theta$ is the angle, so it is defined only up to mod $2\pi n$, and a more natural phase space for my pendulum is the cylinder $\mathbf{S}^1 \times \mathbf{R}$, in this case I have, as it should be expected, only two equilibria, $\hat{x} = 0$ and $\hat{x} = \pi$, which correspond to the lower and upper equilibrium positions of the pendulum respectively).

I can try to figure our stability properties of these equilibria by using the first Lyapunov method (linearization). For this, I find

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}.$$

I have

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

with the eigenvalues $\lambda_{1,2} = \pm \mathrm{i}$, and

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(0,\pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with the eigenvalues $\lambda_{1,2} = \pm 1$ (moreover, the eigenvector corresponding to $+1$ is $(1,1)^\top$, and the eigenvector corresponding to -1 is $(-1,1)^\top$).

So, what do I know exactly from the theorems I proved above? The linearization theorem does not tell me anything about the equilibrium $(0,0)$, because it is the center in the linear system (a non-hyperbolic rest point). The other points are saddles in the linearized system, and therefore, by my main (Lyapunov) theorem, are also unstable in the nonlinear system (the proved theorems do not tell me precisely what is the structure of the orbits of the nonlinear system around these equilibria, but more involved theorems, see Appendix, tell me that it will look approximately like the saddle with the stable and unstable subspaces of the linear case being tangent to some stable and unstable manifolds in the nonlinear case).

Let me now take advantage of Example 4.12. In particular, consider the function

$$E(x,y) = \frac{y^2}{2} - \cos x.$$

A simple computation shows that this function is a first integral of the system, since along the orbits

$$\dot{E} = 0.$$

Therefore, the orbits lay in the level sets of this function. Note that $E(0,0) = -1$, and if I take $V(x,y) = E(x,y) + 1$, then I have that $V(0,0) = 0$, $V(x,y)$ is positive definite anywhere except at the equilibria $(2\pi j, 0)$ of the systems, and $\dot{V} = 0$. Therefore, $V(x,y)$ is a Lyapunov function for $(\hat{x},\hat{y}) = (2\pi j, 0)$, including the most interesting point $(0,0)$, and therefore $(0,0)$ is Lyapunov stable. Is it asymptotically stable? Actually, no, since I have that $\cos x = 1 - \frac{x^2}{2} + \mathcal{O}(|x|^4)$, then

$$V(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + \ldots,$$

where the dots denote terms of the order bigger than 2. Therefore (and if you are not comfortable with this heuristic reasoning you should look up *Morse's lemma*), for x, y close enough to zero, the level sets of V correspond to slightly deformed circles, and the equilibrium $(0,0)$, while being Lyapunov stable, is not asymptotically stable (see the figure below, I will show soon how make such figures without resorting to a computer).

Now introduce damping in the system, so the equation becomes

$$\ddot{\theta} + s\dot{\theta} + k\sin\theta = 0,$$

where $s > 0$ is the constant describing damping.

The system takes the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -sy - \sin x, \end{aligned}$$

with the Jacobi matrix

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\cos x & -s \end{bmatrix}.$$

Note that I have exactly the same equilibria. In the linearized system the equilibrium $(0,0)$ becomes either stable focus ($s < 2$), or stable node ($s > 2$) and therefore in both systems asymptotically stable. The other equilibrium is still a saddle and hence unstable. So, it is quite natural to assume that $(0,0)$ attracts (almost) all the orbits from the cylinder. But I cannot make this conclusion by the linearization technique, which is an essentially *local* tool. It is the Lyapunov function (the first integral in this case), which allowed me to draw the global structure of the phase portrait for the undamped pendulum. Therefore, the next natural thing is to look for a Lyapunov function that would show me something global about the system with damping.

The level sets of the first integral helped to see the structure of the orbits in the system without damping. If, however, V is a strict Lyapunov function, the sets of the form $U_\alpha = \{\boldsymbol{x} \colon V(\boldsymbol{x}) \leq \alpha\}$ can help to see the sets of initial conditions that actually converge to the equilibrium when $t \to \infty$. Let me introduce some terminology.

Definition 4.32. *Let $\hat{\boldsymbol{x}}$ be an asymptotically stable equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Then the basin of attraction of $\hat{\boldsymbol{x}}$, denoted by $B(\hat{\boldsymbol{x}})$, is defined as $B(\hat{\boldsymbol{x}}) = \{\boldsymbol{x}_0 \colon \boldsymbol{x}(t;\boldsymbol{x}_0) \to \hat{\boldsymbol{x}},\, t \to \infty\}$.*

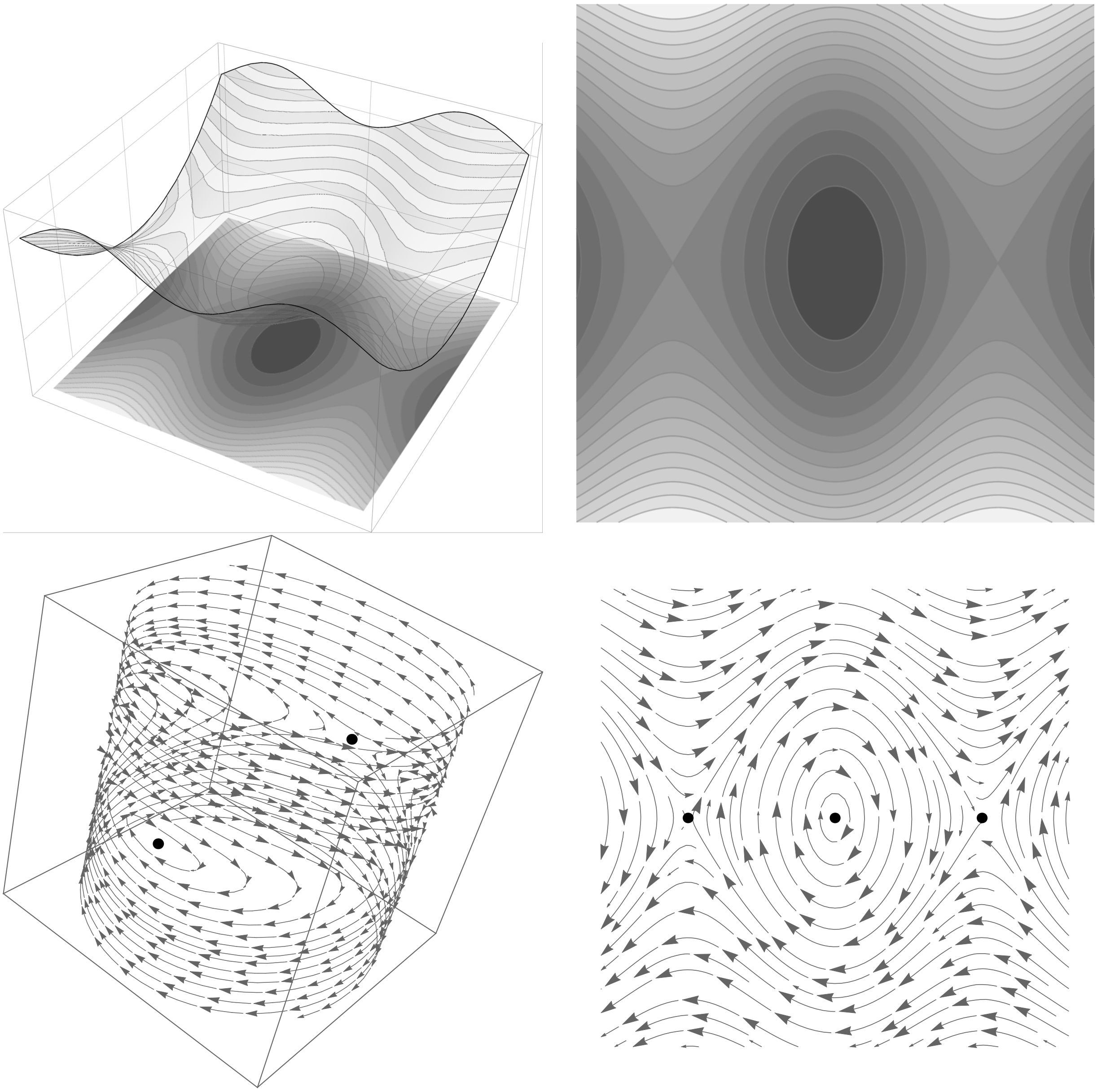

Figure 4.1: Level sets of $V(x,y)=\frac{y^2}{2}+1-\cos x$ in 3D and 2D (the top row), and the phase flow of the system on the cylinder and on the plane (bottom row)

A set $D\subseteq X\subseteq \mathbf{R}^k$ is called invariant with respect to the flow $\{\varphi^t\}$ if for any initial condition $\boldsymbol{x}_0\in D$ the corresponding orbit $\gamma(\boldsymbol{x}_0)\subseteq D$, it is called forward invariant if $\gamma(\boldsymbol{x}_0)$ is replaced with $\gamma_+(\boldsymbol{x}_0)$, the positive semi-orbit through $\boldsymbol{x}_0$, and it is called backward invariant if $\gamma(\boldsymbol{x}_0)$ is replaced with $\gamma_-(\boldsymbol{x}_0)$, the negative semi-orbit.

A forward invariant set that is bounded is called a trapping region.

Using the introduced definitions, I immediately conclude that if V is a strict Lyapunov function, then $U_\alpha \subseteq B(\hat{\boldsymbol{x}})$, and if V is a strict Lyapunov function in U, then it is impossible to have cycles in U (can you formally prove it?). Moreover, if V is strict on all the phase space X then $\hat{\boldsymbol{x}}$ attracts all the orbits, and called *globally asymptotically stable.* If V is a Lyapunov function, then U_α is forward invariant set and if $V(\boldsymbol{x}) \to \infty$ as $|\boldsymbol{x}| \to \infty$ then U_α is a trapping region.

Exercise 4.22. Prove all the statements in the above paragraph.

Now back to the pendulum example. Let me try the same Lyapunov function,

$$V(x,y) = \frac{y^2}{2} + 1 - \cos x.$$

I find that $\dot{V}(x,y) = -sy^2$, which is negative semi-definite, and hence not a strict Lyapunov function. However, note that the set $\dot{V} = 0$ composed of the x-axis, and the only orbit, that starts in this set and stays in this set (it is invariant) is $(0,0)$. Which means that V is "almost" strict, and actually gives me the information about the basin of attraction of $(0,0)$. To formulate these reasoning in a rigorous way, I will need an extra tool that turns out to be very helpful while studying dynamical systems defined by ODE.

Exercise 4.23. Consider the pendulum equation with damping

$$\ddot{\theta} + s\dot{\theta} + \sin\theta = 0, \quad s > 0.$$

I know that the derivative of $V(x,y) = \frac{y^2}{2} + 1 - \cos x$ along the orbits is negative semi-definite, which proves that the origin is Lyapunov stable. On the other hand I also know that the origin is actually asymptotically stable if $s > 0$. Modify V to construct a strict Lyapunov function for $(0,0)$.

Exercise 4.24. Consider the following system

$$x' = 2y^3 - x^5, \quad y' = -x - y^3.$$

Is it globally asymptotically stable?

4.6.2 Limit sets

Definition 4.33. *The ω-limit set of $\boldsymbol{x}_0 \in X$ is*

$$\omega(\boldsymbol{x}_0) = \{\boldsymbol{y} \in X\colon \liminf_{t\to\infty} |\boldsymbol{x}(t;\boldsymbol{x}_0) - \boldsymbol{y}| = 0\}.$$

The α-limit set of $\boldsymbol{x}_0 \in X$ is

$$\alpha(\boldsymbol{x}_0) = \{\boldsymbol{y} \in X\colon \liminf_{t\to-\infty} |\boldsymbol{x}(t;\boldsymbol{x}_0) - \boldsymbol{y}| = 0\}.$$

The definition above means, e.g., for the ω-limit set, that for each $\boldsymbol{y} \in \omega(\boldsymbol{x}_0)$ there exists a sequence $t_1 < t_2 < \ldots < t_n < \ldots$, where $t_n \to \infty$ that the sequence $\big(\boldsymbol{x}(t_n;\boldsymbol{x}_0)\big)$ converges to $\boldsymbol{y}$, and similarly for α-limit set.

Lemma 4.34.

$$\omega(\boldsymbol{x}_0) = \bigcap_{t\in\mathbf{R}} \overline{\gamma_+(\boldsymbol{x}(t;\boldsymbol{x}_0))}, \quad \alpha(\boldsymbol{x}_0) = \bigcap_{t\in\mathbf{R}} \overline{\gamma_-(\boldsymbol{x}(t;\boldsymbol{x}_0))}.$$

Exercise 4.25. Prove this lemma.

Theorem 4.35. *The limit sets are closed and invariant. If $\gamma_+(\boldsymbol{x}_0)$ is bounded then $\omega(\boldsymbol{x}_0)$ is non empty and connected. If $\gamma_-(\boldsymbol{x}_0)$ is bounded then $\alpha(\boldsymbol{x}_0)$ is non empty and connected.*

Exercise 4.26. Prove Theorem 4.35.

Exercise 4.27. Can you give an example of an orbit which has a non-empty disconnected omega limit set?

Exercise 4.28. Consider the system

$$\dot{r} = r(a - r), \quad \dot{\theta} = b, \quad a > 0, b \in \mathbf{R},$$

where (r, θ) are polar coordinates in the plane. Find the omega limit sets for any initial condition on the plane.

Answer the same question for

$$\dot{r} = r(a - r), \quad \dot{\theta} = \sin^{\theta} + (r - a)^2, \quad a > 0.$$

Exercise 4.29. A gradient system is the system of the form

$$\dot{\boldsymbol{x}} = -\operatorname{grad} V(\boldsymbol{x}) = -\nabla V(\boldsymbol{x}),$$

where $V \colon X \longrightarrow \mathbf{R}$ is a $\mathcal{C}^{(2)}$ function. Prove

Theorem 4.36. *Let $\hat{\boldsymbol{x}}$ be an isolated minimum of V. Then $\hat{\boldsymbol{x}}$ is an asymptotically stable equilibrium of the gradient system.*

Theorem 4.37. *Let $\boldsymbol{y}$ be an α or ω limit point of a solution to the gradient system. Then $\boldsymbol{y}$ is an equilibrium.*

Using the notions of the limit sets I shall prove the so-called *invariance* principle, which is usually attributed to Krassovkii and LaSalle[1].

Theorem 4.38 (Invariance principle)**.** *Let V be continuously differentiable, $U = \{\boldsymbol{x} \in X \colon V(\boldsymbol{x}) < \alpha\}$ for some real number α and V be continuous on ∂U. Let $\dot{V}(\boldsymbol{x}) \leq 0$ for $\boldsymbol{x} \in U$. Let $Q = \{\boldsymbol{x} \in U \colon \dot{V}(\boldsymbol{x}) = 0\}$ and let M be the largest invariant set in Q. Then every positive orbit that starts in U and remains bounded has its ω-limit set in M.*

Proof. Let $\boldsymbol{x}_0 \in U$ and $\gamma_+(\boldsymbol{x}_0)$ be bounded. Then the fact that $\dot{V}(\boldsymbol{x}) \leq 0$ and the definition of U imply that $\gamma_+(\boldsymbol{x}_0) \in U$, i.e., U is positive invariant. Consequently, there exists a limit $V(\boldsymbol{x}(t; \boldsymbol{x}_0)) = \beta$ when $t \to \infty$. The continuity of V implies that $V(\boldsymbol{y}) = \beta$ for any $\boldsymbol{y} \in \omega(\boldsymbol{x}_0)$. Since $\omega(\boldsymbol{x}_0)$ is invariant then $V(\boldsymbol{x}(t; \boldsymbol{y})) = \beta$ for all $t \in \mathbf{R}$, therefore $\omega(\boldsymbol{x}_0) \subseteq Q$ and $\omega(\boldsymbol{x}_0) \subseteq M$. ■

Corollary 4.39. *Let $V \colon U \longrightarrow \mathbf{R}$ be a Lyapunov function for $\hat{\boldsymbol{x}}$, U be positive invariant, and $M \subseteq U$ consists only of $\hat{\boldsymbol{x}}$. Then $\hat{\boldsymbol{x}}$ is asymptotically stable, and $U \subset B(\hat{\boldsymbol{x}})$.*

[1] A special case of this principle was originally published by Barbashin and Krasovskii in 1952, the full version by Krassovskii in 1959 and, independently, by LaSalle in 1960

Let me apply this corollary to the pendulum with damping. I calculated that for

$$V(x, y) = \frac{y^2}{2} + 1 - \cos x,$$

the derivative with respect to the vector field is given by

$$\dot{V}(x, y) = -2sy^2.$$

The set Q in this case is given by $(x, 0)$ for any $x \in (-\pi, \pi)$ if I consider the cylinder as the phase space. The half axis $y = 0, x > 0$ is not invariant under the flow of the system, and therefore the only invariant set in this case $M = (0, 0)$, which, according to the corollary above, is therefore asymptotically stable, and (almost) any orbit on the cylinder converges to this equilibrium as $t \to \infty$.

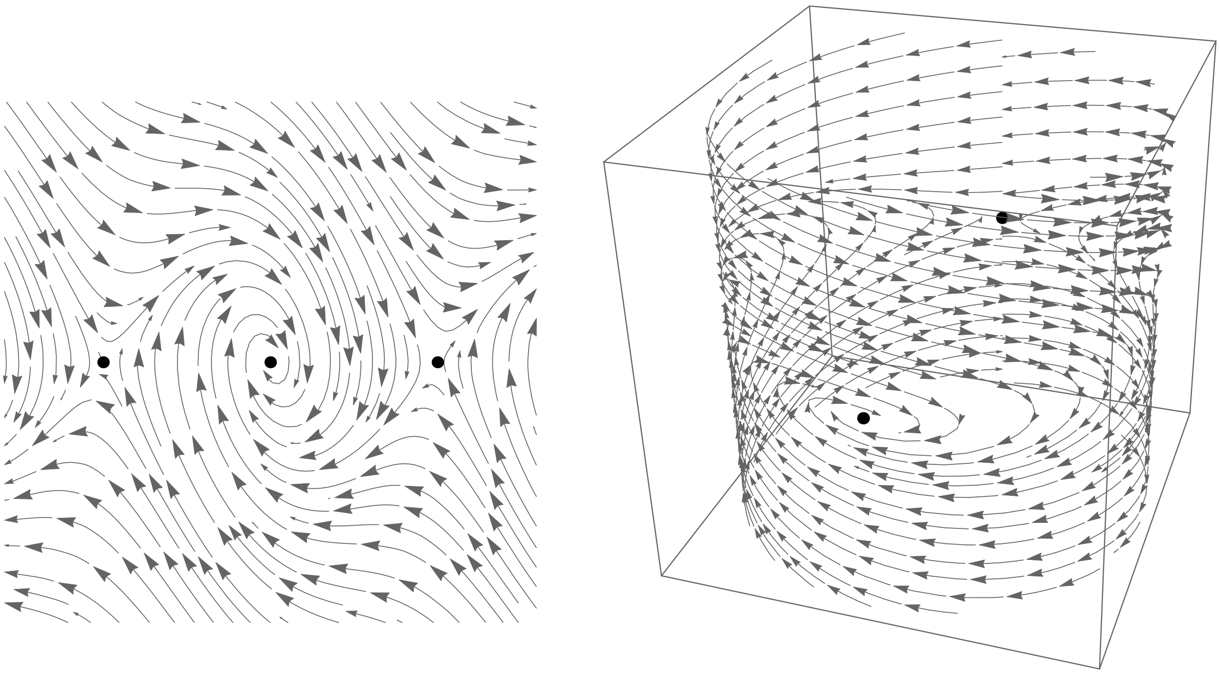

Figure 4.2: The phase portrait of the pendulum with damping on the plane and on the cylinder

Exercise 4.30. Determine the stability properties of the origin for

$$\begin{aligned} \dot{x} &= -x^3 + 2y^3, \\ \dot{y} &= -2xy^2. \end{aligned}$$

Hint: Start with $V(x, y) = x^2 + cy^2$ and determine c.

Exercise 4.31. Determine the stability properties of the origin for

$$\begin{aligned} \dot{x} &= -y - x^3, \\ \dot{y} &= x^5. \end{aligned}$$

4.7 One dimensional movement of a particle in a potential field

As I mentioned in Example 4.12 a one dimensional movement of a partial in a potential field is described by the second Newton's law

$$m\ddot{x} = -U'(x). \tag{4.8}$$

This equation is equivalent to the system

$$\dot{x} = y, \quad m\dot{y} = -U'(x),$$

which has the first integral

$$\frac{my^2}{2} + U(x) = E,$$

the physical interpretation of this relation is the law of the total energy, which is equal to the sum of kinetic and potential energies. From the first integral I immediately get

Corollary 4.40. *Equation* (4.8) *can be integrated as*

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - U(x))} \Longrightarrow \pm\sqrt{\frac{m}{2}}\int_{x_2}^{x} \frac{\mathrm{d}s}{\sqrt{E - U(s)}} = t - t_0,$$

where the sign is chosen according to the initial velocity of the particle. Here $E = K + U$ is the constant total energy.

The last corollary actually implies that the graph of U is all what I need to sketch the phase plane of the second order autonomous system. I will show how to do this be way of an example. Let me assume that the potential U looks like in Figure 4.3, top panel. So I assume that $U \in \mathcal{C}^{(2)}$ and there are only two points $\hat{x}_1$ and $\hat{x}_2$ at which $U'(x) = 0$, I also assume that $U(-\infty) = \infty$, $U(\infty) = -\infty$. Due to the form of the first integral I must have that $U(x) \leq E$, and hence the movement with the given full energy E can happen only when $U(x) \leq E$; in my specific example this corresponds to the intervals (x_1, x_2) and (x_2, ∞). At the points where $U(x) = E$ I have that $\dot{x} = 0$, that is the velocity of the particle is equal to zero, these points are called the stop points.

Assume that initially $x_0 \in [x_1, x_2]$ and therefor it will belong to the interval for all times $t \in \mathbf{R}$. If $\dot{x}(0) > 0$ then initially I have the movement from lent to right, such that

$$\sqrt{\frac{m}{2}}\int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{E - U(x)}} = t$$

for small enough $t > 0$. At the moment

$$t_1 = \sqrt{\frac{m}{2}}\int_{x_0}^{x_2} \frac{\mathrm{d}x}{\sqrt{E - U(x)}}$$

my particle will arrive at the point x_2. Note that $t_1 < \infty$ since $U'(x_2) \neq 0$ and hence $E - U(x) \sim -U'(x_2)(x - x_2)$ for $x \to x_2$ and the integral converges. This means that the particle turns left and now her movement is described by

$$t = t_1 - \sqrt{\frac{m}{2}}\int_{x_2}^{x} \frac{\mathrm{d}x}{\sqrt{E - U(x)}}$$

until it arrives to x_1. The period T of the particle oscillations equals twice the time required to move from x_1 to x_2, that is

$$T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{E - U(x)}} ,$$

where $U(x_i(E)) = E$, $i = 1, 2$.

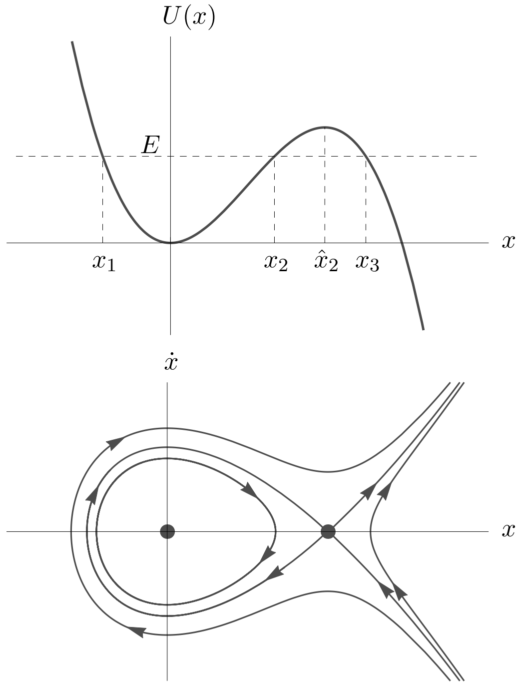

Figure 4.3: Inferring the phase portrait from the potential function

If the initial position of the particle with energy E is to the right of x_3 and $\dot{x}(0) < 0$. Then, analogously, the particle fist will move left up to the point x_3, turns right, and now goes to the right without turning back (this is called an infinite movement opposite to the oscillations with correspond to the finite movement).

Let $E_0 = U(\hat{x_2})$. Then if $x_0 < \hat{x}_2$ then the particle will travel through the homoclinic separatrix of the saddle $\hat{x}_2$ (what is time required to go through the whole separatrix?), if $x_0 > \hat{x}_2$ then the movement is again infinite. Therefore, separatrices separate finite from infinite movements in this model.

The analysis of the example above implies that it is possible to have only centers and saddles in the mechanical systems described by the Newton law (4.8), moreover the centers correspond to the minima and the saddles to the maxima of the potential function.

Exercise 4.32. Consider the pendulum equation

$$\ddot{\theta} + \sin\theta = 0,$$

with the initial conditions $\theta(0) = a > 0$, $\dot{\theta}(0) = 0$. Show that the period of the solution can be found as

$$T = 4\int_0^a \frac{\mathrm{d}\theta}{(2\cos\theta - 2\cos a)^{1/2}}\,.$$

Use the last expression to show that

$$T = 2\pi\left(1 + \frac{a^2}{16} + \mathcal{O}(a^4)\right).$$

Is this solution Lyapunov stable?

Exercise 4.33. Consider the equation

$$\ddot{x} + 2a\dot{x} + x + x^3 = 0, \quad 0 < a < 1.$$

If $a = 0$ then the system is conservative. Find its first integral V. If $a \neq 0$ then the origin is asymptotically stable, prove it using the linearization. Use V as a Lyapunov function for the equation to show that the origin is stable and use the invariance principle to prove that it is asymptotically stable. Use the specific form of V to find the basin of attraction of the origin and argue that it is $\mathbf{R}^2$ (such systems, whose basin of attraction coincides with the whole state space, are called *globally asymptotically stable*).

4.8 Appendix

4.8.1 Perron's theorem

4.8.2 Stability of periodic solutions and other notions of stability

4.8.3 Big theorem of ODE theory

4.8.4 Classical mechanics with one degree on freedom

4.8.5 Replicator equation and mathematical biology